A THEORY OF LENGTH FOR NOETHERIAN MODULES

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Introduction.

In this paper we shall introduce a theory of length for Noetherian modules over an arbitrary ring (with identity), assigning to each Noetherian module M an ordinal number $l(M)$ which will briefly be called the length of M, see § 2 for definition. $l(M)$ is finite if and only if M has a finite composition-series, in which case $l(M)$ equals the length of the composition-series. Thus we are working with a generalization of the classical theory of length.

$l(M)$ carries important information about M. Being an ordinal, $l(M)$ can be expressed as a polynomial in ω with integral coefficients and ordinal exponents, ω denoting the first non-finite ordinal. This polynomial - the Cantor normal form of $l(M)$ - has properties similar to the properties of the Hilbert-Samuel polynomials in local algebra. First of all, its degree coincides with the Krull dimension of M (2.3), the Krull dimension being interpreted as an ordinal as in Krause [5]. Moreover, if α is an ordinal, then the coefficient of the term of degree α is additive on the category of Noetherian modules of Krull dimension not greater than α (2.7).

In § 1 we fix the notation concerning ordinal numbers and the Krull ordinal of a partially ordered Noetherian set.

§ 2 contains general results concerning the length function $M \mapsto l(M)$. Although l is not additive in general, 2.1 gives the following satisfactory substitute for additivity: if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of Noetherian modules, then we have

$$l(M'') + l(M') \leq l(M) \leq l(M') \oplus l(M'')$$

Moreover we have (2.11):

$$l(M' \oplus M'') = l(M') \oplus l(M'').$$

Here $\oplus$ is used ambiguously to denote the Hessenberg natural sum of ordinals, cf. § 1, and the direct sum of modules.

In general there does not exist a good notion of composition series in terms of which $l(M)$ can be defined. However, we show in 2.12 that if M has countable Krull dimension, then there exists a chain of non-zero submodules of M which is of ordinal type $l(M)$.

Unlike the case with factor modules of M (2.3), not every ordinal less than $l(M)$ is the length of a submodule of M. In fact if N is a submodule of M then each of the coefficients in the polynomial $l(N)$ is less than or equal to the corresponding coefficient in the polynomial $l(M)$. In particular, $l(N)$ can only take a finite number of values (2.9).

In § 3 we obtain more precise results by assuming that all modules be finitely generated over a commutative Noetherian ring. In this case we can give an interpretation of the set of exponents in the polynomial $l(M)$, in terms of $\operatorname{Ass} M$ (3.2). We also give a complete description of the possible lengths of the submodules of M.

In Bass [1] $o(M)$ denotes the supremum of the ordinal types of descending chains of non-zero submodules of M. In 3.4 we show that also $o(M)$ can be expressed in terms of $l(M)$. We have the relation

$$o(M) = \min(\omega_1, l(M))$$

ω_1 being the first non-countable ordinal.

§ 1 Notation and basic definitions.

If W is a set of ordinal numbers, we let $\sup W$ denote the least ordinal which is greater than or equal to every element in W. In particular we put $\sup \emptyset = 0$. If $\beta_1, \cdots, \beta_k$ are ordinals, we let $\sum_{i=1}^{k} \beta_i$ denote their sum in the following order

$$\beta_1 + \cdots + \beta_k$$

Letting ω denote the ordinal type of the natural numbers, any ordinal α can be written

$$* \qquad \alpha = \sum_{i=1}^{k} \omega^{\alpha_i} n_i$$

where $n_1, \cdots, n_k$ are non-negative integers and the exponents form a decreasing sequence of ordinals, i.e.

$$j < i \Rightarrow \alpha_i < \alpha_j \qquad \text{for all } i,j$$

The representation (*) will be called the Cantor normal form of α. If $n_1 \neq 0$ the corresponding exponent α_1 will be called the <u>degree</u> of α and will be denoted by $\deg \alpha$. It is convenient to define $\deg 0 = -1$. The Cantor normal form is unique in the following sense: Let α and β be ordinals with Cantor normal forms $\sum_{i=1}^{k} \omega^{\alpha_i} n_i$ and $\sum_{i=1}^{k} \omega^{\alpha_i} m_i$ respectively. Then we have $\alpha = \beta$ if and only if $n_i = m_i$ for all i. If $n_i \leq m_i$ for all i, then this fact will be expressed by writing $\alpha << \beta$. Finally we define the <u>direct sum</u> (Hessenberg natural sum) of α and β as follows

$$\alpha \oplus \beta := \sum_{i=1}^{k} \omega^{\alpha_i} (n_i + m_i).$$

A justification for this notation is contained in 2.11.

Let S be a non-empty partially ordered set which is Noetherian, i.e. every non-empty subset has a maximal element. Let $\underline{\mathrm{Ord}}$ denote the class of ordinal numbers. By the ordinal map on S we mean the map

$$\lambda: S \to \underline{\mathrm{Ord}}$$

defined by

$$\lambda(x) = \sup\{\lambda(y) + 1 : \ x < y\}$$

The Krull ordinal of S will be denoted $\varkappa(S)$ as in [1]. $\varkappa(S)$ can be expressed in terms of the ordinal map as follows

$$\varkappa(S) = \sup\{\lambda(x): \ x \in S\}.$$

§ 2 The length of Noetherian modules.

Let M be a Noetherian (left)module over a ring (with identity) and let $S(M)$ be the set of all submodules of M ordered by inclusion. The Krull ordinal of $S(M)$ will be called the length of M and will be denoted by $l(M)$. The degree of the ordinal $l(M)$, cf. § 1, will be called the dimension of M and will be denoted $d(M)$. By the Krull dimension of M we will mean the ordinal $\mathrm{Kdim}M$ as defined in Krause [5] and equivalently in [2]. We shall see in theorem 2.3 below that $d(M) = \mathrm{Kdim}\,M$.

2.1 Theorem. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Notherian modules. Then we have

$$l(M'') + l(M') \leq l(M) \leq l(M') \oplus l(M'')$$

In particular we have

$$d(M) = \max(d(M'), d(M'')).$$

<u>Proof:</u> The last equality clearly follows from the two inequalities. We will start by proving the first inequality. Let P be the partially ordered set obtained from $S(M')$ and $S(M'')$ by identifying the unique maximal element in $S(M')$ with the unique minimal element in $S(M'')$. Let λ^+ and λ' be the ordinal maps on P and $S(M')$ respectively. It is easily shown by induction that

$$\lambda^+(N) = \varkappa(S(M'')) + \lambda'(N) \qquad \text{for all } N \in S(M')$$

Hence

$$\varkappa(P) = \varkappa(S(M'')) + \varkappa(S(M')) = l(M'') + l(M').$$

Since we have an order preserving injection $P \to S(M)$, it is easily shown that $\varkappa(P) \leq \varkappa(S(M))$. Hence

$$l(M'') + l(M') \leq l(M).$$

We shall now prove the second inequality in 2.1. Let λ'', λ and λ' denote the ordinal maps on $S(M'')$, $S(M)$ and $S(M')$ respectively. We will define a map

$$\lambda^* : S(M) \to \underline{\mathrm{Ord}}$$

as follows. Let $N \in S(M)$. Put

$$\lambda^*(N) = \lambda'(N \cap M') \oplus \lambda''(N+M'/M')$$

I claim that λ^* is strictly order reversing. Indeed, let $N_1 \subseteq N_2$ be submodules of M. Clearly we have

$$\lambda^*(N_1) \geq \lambda^*(N_2)$$

Assume that we have equality. We are going to show that $N_1 = N_2$. For $i = 1,2$ put

$$\alpha_i = \lambda'(N_i \cap M') \quad \text{and} \quad \beta_i = \lambda''(N_i + M'/M')$$

We have

$$\alpha_1 \geq \alpha_2 \ , \quad \beta_1 \geq \beta_2 \text{ and } \alpha_1 \oplus \beta_1 = \alpha_2 \oplus \beta_2$$

These three relations are easily seen to imply

$$\alpha_1 = \alpha_2 \quad \text{and} \quad \beta_1 = \beta_2 .$$

Since λ' and λ'' are strictly order reversing we have

$$N_1 \cap M' = N_2 \cap M' \quad \text{and} \quad N_1 + M'/M' = N_2 + M'/M'.$$

It follows that $N_1 = N_2$. Since λ^* is strictly orderreversing, it is easily shown by induction that

$$\lambda(N) \leq \lambda^*(N) \quad \text{for all} \quad N \in S(M).$$

Hence

$$l(M) = \varkappa(S(M)) = \lambda((o)) \leq \lambda^*((o)) = l(M') \oplus l(M'')$$

2.2 <u>Remark.</u> It is possible to generalize the notion of length to non-Noetherian modules M, by letting $l(M)$ be the supremum of all ordinals $\varkappa(S)$ where S runs through the set of all Noetherian subsets of $S(M)$. With this generalized notion the previous theorem would still be valid, except for the first of the two inequalities which has to be replaced by the following weaker inequality

$$\max(l(M''), l(M')) \leq l(M).$$

2.3 <u>Theorem.</u> Let M be a non-zero Noetherian module. Then we have

(i) Every ordinal less than $l(M)$ is the length of a proper factor module of M. Conversely, if N is a non-zero submodule of M then $l(M/N) < l(M)$.

(ii) $d(M) = \operatorname{Kdim} M$.

<u>Proof:</u> (i) Let β be an ordinal less than $l(M)$, and let λ be the ordinal map on $S(M)$. Letting 0_M denote the zero-submodule in M we have $\lambda(0_M) = l(M) > \beta$. Hence we can find a submodule $N \subseteq M$ such that $\lambda(N) = \beta$, so $l(M/N) = \beta$. Conversely, if N is a non-zero submodule of M, then by 2.1 $l(M/N) < l(M)$.

(ii) We will first show that $\operatorname{Kdim} M \leq d(M)$ using induction on $d(M)$. If $l(M) \leq 0$ then M has finite length, so clearly $\operatorname{Kdim} M = d(M)$. Let α be a non-zero ordinal and assume that the inequality is valid whenever $d(M) < \alpha$. Now assume that $d(M) = \alpha$. Assume that $\operatorname{Kdim} M > \alpha$. Then there exists a descending chain

$$M = M_0 \supset M_1 \supset \cdots$$

such that $\operatorname{Kdim} M\, M_i/M_{i+1} \geq \alpha$ for $i \geq 0$. By the induction hypothesis we have $d(M_i/M_{i+1}) \geq \alpha$. Hence $l(M_i/M_{i+1}) \geq \omega^\alpha$. By 2.1 we have $l(M) \geq \omega^\alpha \omega = \omega^{\alpha+1}$. So $d(M) \geq \alpha+1$ which is a contradiction. We conclude that $\operatorname{Kdim} M \leq \alpha$.

We will now show that $d(M) \leq \operatorname{Kdim} M$ using induction on $\operatorname{Kdim} M$. If $\operatorname{Kdim} M \leq 0$ then M has finite length, hence $d(M) = \operatorname{Kdim} M$. Put $\operatorname{Kdim} M = \alpha > 0$. Assume that $d(M) \geq \alpha+1$. Then $l(M) \geq \omega^{\alpha+1}$.
By (i) we can find a submodule $M_1 \subset M$ such that $l(M/M_1) = \omega^\alpha$. By 2.1 it follows that

$$\omega^{\alpha+1} \leq l(M) \leq l(M_1) \oplus \omega^\alpha$$

Hence $l(M_1) \geq \omega^{\alpha+1}$. Now we can find a submodule $M_2 \subset M_1$ such that $l(M_1/M_2) = \omega^\alpha$. Repeating the argument we can find a descending sequence $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ such that $d(M_i/M_{i+1}) = \omega^\alpha$ for $i \geq 0$. Hence $d(M_i/M_{i+1}) = \alpha$. We may assume by induction that

$\operatorname{Kdim}(M_i/M_{i+1}) \geq \alpha$. Hence $\operatorname{Kdim} M \geq \alpha+1$, which is a contradiction. We conclude that $d(M) \leq \alpha$. ∎

2.4 <u>Corollary</u>. To each ordinal α there exists a Noetherian, commutative ring R such that $l(R) = \alpha$.

<u>Proof</u> There exists a commutative, Noetherian ring R_α such that $\operatorname{Kdim} R_\alpha \geq \alpha$, cf. [2] or [3]. Hence $l(R_\alpha) \geq \omega^\alpha \geq \alpha$. By 2.3(i) there exists an ideal $\mathfrak{A}$ in R_α such that $l(R_\alpha/\mathfrak{A}) = \alpha$. ∎

In [2] a module M is called α-critical if M has Krull-dimension equal to α and every proper factor-module has Krull-dimension less that α . The following corollary is an immediate consequence of 2.3:

2.5 <u>Corollary</u>. Let M be a Noetherian module. Then the following statements are equivalent:

(i) M is α-critical.

(ii) $l(M) = \omega^\alpha$.

2.6 <u>Definition</u> Let M be a Noetherian module and let α be any ordinal. The coefficient of the term of degree α in the Cantor normal form of $l(M)$ is a non-negative integer which will be denoted by $\mu_\alpha(M)$.

2.7 <u>Lemma</u> Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of Noetherian modules. Put $\alpha = \operatorname{Kdim} M$. Then we have

$$\mu_\alpha(M) = \mu_\alpha(M') + \mu_\alpha(M'')$$

<u>Proof</u> By 2.3 α equals the degree of $l(M)$, hence the equality follows from 2.1. ∎

2.8 Lemma and definition. Let M be a Noetherian module of dimension $\alpha \neq 0$. Then there exists a unique maximal submodule of M of dimension less that α, which will be denoted by M_*. Put

$$l(M) = \omega^\alpha n + \beta$$

where $n \neq 0$ and $\beta < \omega^\alpha$ then we have $l(M_*) = \beta$ and $l(M/M_*) = \omega^\alpha n$.

Proof Since M is Noetherian, the existence of M_* is clear in view of 2.1. By 2.3 we can choose a submodule N in M such that $l(M/N) = \omega^\alpha n$. Using 2.6 we obtain $\mu_\alpha(N) = 0$, hence $\mathrm{Kdim}\, N < \alpha$, so $N \subseteq M_*$. Moreover it follows from 2.1 that

$$\omega^\alpha n + l(N) \leq l(M) \leq \omega^\alpha n \oplus l(N) = \omega^\alpha n + l(N)$$

Hence

$$\omega^\alpha n + l(N) = l(M) = \omega^\alpha n + \beta$$

so $l(N) = \beta$. It suffices to show that $N = M_*$. Since

$$\mu_\alpha(M/M_*) = \mu_\alpha(M)$$

we have

$$l(M/M_*) = \omega^\alpha n + \gamma$$

for some γ. Using 2.1 on the exact sequence

$$0 \to {}^{M_*}/_N \to {}^{M}/_N \to {}^{M}/_{M_*} \to 0$$

we obtain

$$\omega^\alpha n + \gamma + l(M_*/N) \leq l(M/N) = \omega^\alpha n .$$

Hence we have $l(M_*/N) = 0$ so $M_* = N$. ∎

2.9 Theorem. Let M be a Noetherian module and consider the following sets of ordinals:

$$A(M) := \{\beta : l(M) = \gamma + \beta \text{ for some ordinal } \gamma\}$$

$$lS(M) := \{l(N) : N \subseteq M\}$$

$$C(M) := \{\beta : \beta << l(M)\}$$

Then we have

$$A(M) \subseteq lS(M) \subseteq C(M) .$$

<u>Proof</u> We will first prove that $A(M) \subseteq lS(M)$. Let γ and β be ordinals such that $l(M) = \gamma + \beta$. We are going to show the existence of a submodule $N \subseteq M$ such that $l(N) = \beta$.
Let α be the degree of β . We may write

$$l(M) = \gamma' + \omega^{\alpha}m + \beta'$$

where $\deg \beta' < \alpha$ and where each term in the Cantor normal form of γ' has degree greater than α . Clearly there exists an integer $n \leq m$ such that

$$\beta = \omega^{\alpha}n + \beta'$$

By repeated application of the operation $*$ in 2.8 we obtain a submodule $N_1 \subseteq M$ such that

$$l(N_1) = \omega^{\alpha}m + \beta'$$

By 2.3 we can find a submodule $N \subseteq N_1$ such that

$$l(N_1/N) = \omega^{\alpha}(m-n)$$

Using 2.1 on the exact sequence

$$0 \to N \to N_1 \to N_1/N \to 0$$

we obtain

$$\omega^{\alpha}(m-n) + l(N) \leq l(N_1) \leq \omega^{\alpha}(m-n) \oplus l(N) = \omega^{\alpha}(m-n) + l(N)$$

Hence

$$\omega^{\alpha}(m-n) + l(N) = l(N_1) = \omega^{\alpha}m + \beta'$$

So

$$l(N) = \omega^{\alpha}n + \beta' = \beta \ .$$

To prove the relation $lS(M) \subseteq C(M)$, let N be any submodule of M . We are going to show that $l(N) << l(M)$, i.e. $\mu_\alpha(N) \leq \mu_\alpha(M)$ for all α . We will use induction on the dimension of M . If $\operatorname{Kdim} M = 0$, then N and M have finite length, and the inequalities are satisfied in this case.
We will now assume that $\operatorname{Kdim} M > 0$. By the obvious induction hypotesis it follows that

(1) $\qquad \mu_\alpha(N \cap M_*) \leq \mu_\alpha(M_*) \qquad$ for all α .

Moreover, it follows from 2.7 that

(2) $\qquad \mu_\alpha(M_*) = \mu_\alpha(M) \qquad$ for all $\alpha \neq \operatorname{Kdim} M$

(3) $\qquad \mu_\alpha(M_*) = 0 \qquad$ for $\alpha = \operatorname{Kdim} M$.

There are two cases:

(i) $\operatorname{Kdim} N < \operatorname{Kdim} M$. In this case we have $N = N \cap M_*$.
Hence by (1), (2) and (3) we have

$$\mu_\alpha(N) \leq \mu_\alpha(M) \qquad \text{for all } \alpha \ .$$

(ii) $\operatorname{Kdim} N = \operatorname{Kdim} M$. In this case we have $N_* = N \cap M_*$.
For $\alpha \neq \operatorname{Kdim} M$ we have

$$\mu_\alpha(N) = \mu_\alpha(N_*) \leq \mu_\alpha(M_*) = \mu_\alpha(M)$$

For $\alpha = \operatorname{Kdim} M$ it follows from 2.7 that

$$\mu_\alpha(N) = \mu_\alpha(M) - \mu_\alpha(M/N) \leq \mu_\alpha(M)$$

2.10 Remark Jategaonkar shows in [4] that, given any ordinal α, there is a principal right ideal domain R whose proper right ideals are linearly ordered of order type ω^α. Considering R as a right module it is easily seen that we have $A(R) = lS(R)$. In 3.2 below we shall see that if M is a Noetherian module over a commutative ring, then we have $lS(M) = C(M)$. This, combined with 2.4, shows that $A(M)$ is not equal to $lS(M)$ in general.

The inclusion $lS(M) \subseteq C(M)$ expresses that if N is a submodule of M, then each of the coefficients in the Cantor normal form of $l(N)$ is less that or equal to the corresponding coefficient in the Cantor normal form of $l(M)$. This will be referred to as the principle of coefficientwise comparison.

2.11 Proposition Let M be a Noetherian module, and let M_1 and M_2 be submodules such that $M = M_1 + M_2$. Then the sum is direct if and only if

$$l(M) = l(M_1) \oplus l(M_2)$$

Proof We will first show that

$$l(M') \oplus l(M'') = l(M' \oplus M'')$$

The inequality $\geq$ follows immediately from 2.1. We are going to show the opposite inequality by induction on $l(M'')$. For $l(M'') = 0$ there is nothing to prove. Now let $l(M'') > 0$, and let $l(M')$, $l(M'')$ and $l(M' \oplus M'')$ be denoted by α', α'' and α respectively. Letting β be a variable running over the ordinals less that α'' we have $\alpha'' = \sup(\beta+1)$. For each value of β we can find (2.3) a proper factor module $\bar{M}''$ of M'' such that $l(\bar{M}'') = \beta$. Since $M' \oplus \bar{M}''$ is a proper factor-module of $M' \oplus M''$, it follows by the obvious induction hypotesis that

$$l(M') \oplus l(\bar{M}'') \leq l(M' \oplus \bar{M}'') < l(M' \oplus M'') = \alpha$$

Hence

$$(\alpha' \oplus \beta) + 1 \leq \alpha$$

This gives

$$\alpha' \oplus \alpha'' = \alpha' \oplus (\sup(\beta+1)) = \sup((\alpha' \oplus \beta)+1) \leq \alpha ,$$

which was to be shown.

Let us now assume that

$$l(M) = l(M_1) \oplus l(M_2)$$

It remains to show that $M_1 \cap M_2 = 0$. We have an exact sequence

$$0 \to M' \cap M'' \to M' \oplus M'' \to M \to 0$$

Using 2.1 we obtain

$$l(M) + l(M' \cap M'') \leq l(M' \oplus M'') = l(M') \oplus l(M'') = l(M)$$

Hence $l(M' \cap M'') = 0$ so $M' \cap M'' = 0$. ∎

2.12 <u>Proposition</u> Let M be a Noetherian module. Assume that $\mathrm{Kdim}\, M$ is countable. Then there exists a well ordered chain of non-zero submodules of M of ordinal type equal to $l(M)$.

<u>Proof</u> We will use induction on $l(M)$. Put $\alpha := \mathrm{Kdim}\, M$. If $l(M)$ is finite, then the proposition is obvious. Hence we may assume that $\alpha \geq 1$. We will first treat the case where $l(M) = \omega^{\alpha}$. Since α is countable, we can find a non-decreasing sequence of ordinal numbers less than α

$$\beta_1 \leq \beta_2 \leq \ldots \leq \beta_n \leq \ldots$$

such that

$$\omega^{\alpha} = \omega^{\beta_1} + \omega^{\beta_2} + \ldots + \omega^{\beta_n} + \ldots$$

We are going to construct a filtration of non-zero submodules

$$M = M_0 \supset M_1 \supset M_2 \supset \dots$$

such that

$$l(M_{i-1}/M_i) = \omega^{\beta_i} \qquad \text{for } i \geq 1 .$$

We put $M_0 := M$. Now let $i \geq 1$ and assume that $M_0, \dots, M_{i-1}$ has been constructed. By the principle of coefficientwise comparison (2.10) any non-zero submodule of M has length equal to ω^α, hence

$$l(M_{i-1}) > \omega^{\beta_i}$$

Thus by 2.3 we can find a non-zero submodule $M_i \subset M_{i-1}$ such that

$$l(M_{i-1}/M_i) = \omega^{\beta_i} ,$$

and the construction is complete.

By the induction hypotesis M_{i-1}/M_i contains a chain consisting of non-zero submodules and having ordinal type equal to ω^{β_i}. Clearly these chains induce a chain in M of ordinal type ω^α.

In the general case we can write

$$l(M) = \omega^\alpha n + \beta$$

where $n \neq 0$ and $\beta < \omega^\alpha$. By the first part of the proof we may assume that $l(M) > \omega^\alpha$. By 2.3 we can find a non-zero submodule $N \subset M$ such that $l(M/N) = \omega^\alpha$. Using 2.1 on the exact sequence

$$0 \to N \to M \to M/N \to 0$$

we obtain

$$\omega^\alpha + l(N) \leq l(M) \leq \omega^\alpha \oplus l(N) = \omega^\alpha + l(N)$$

Hence

$$l(M) = \omega^\alpha + l(N) .$$

By the induction hypotesis, M/N and N contain chains of ordinal type ω^α and $l(N)$ respectivily. Two such chains clearly induce a chain in M of ordinal type $l(M)$. ∎

§ 3 Noetherian modules over commutative rings.

In this section all modules are assumed to be finitely generated over a commutative Noetherian ring R . The results depend heavily on the assumption that R be commutative.

3.1 Lemma Let M be a module with length

$$l(M) = \omega^{\alpha}n+\gamma$$

where $n \neq 0$ is a natural nimber and $\gamma < \omega^{\alpha}$. Let k be an integer such that $0 \leq k \leq n$. Then M contains a submodule N such that $l(N) = \omega^{\alpha}k$.

Proof By ascending induction on k we are going to construct submodules

$$0 = N_0 \subset \ldots \subset N_k \subset \ldots \subset N_n$$

such that $l(N_k) = \omega^{\alpha}k$. Assume that $1 \leq k \leq n$ and that $N_o, \ldots, N_{k-1}$ has been constucted. By 2.7 we have

$$\mu_{\alpha}(M/N_{k-1}) = (n-k+1) \neq 0$$

Hence $\mathrm{Kdim}\, M/N_{k-1} = \alpha$, so there exists a prime ideal $\mathfrak{p}$ in $\mathrm{Ass}(M/N_{k-1})$ such that $\mathrm{Kdim}\, R/\mathfrak{p} = \alpha$. In view of 2.5 we have $l(R/\mathfrak{p}) = \omega^{\alpha}$. There exists an injection of $R/\mathfrak{p}$ into M/N_{k-1}. The image of $R/\mathfrak{p}$ in M/N_{k-1} pulls back to a submodule in M which we will denote by N_k . Thus we have an exact sequence

$$0 \to N_{k-1} \to N_k \to R/\mathfrak{p} \to 0$$

By (2.1) we obtain $l(N_k) = l(N_{k-1}) + l(R/\mathfrak{p}) = \omega^{\alpha}k$. ∎

3.2 Theorem Let M be a Noetherian module over a commutative ring R , and let the length $l(M)$ have Cantor normal form

$$l(M) = \omega^{\alpha_k} n_k + \ldots + \omega^{\alpha_1} n_1$$

where $n_1 \ldots n_k \neq 0$. Then

(i) M is an essential extension of a direct sum of submodules N_i such that $l(N_i) = \omega^{\alpha_i} n_i$ $(1 \leq i \leq k)$.

(ii) $\{l(N) : N \subseteq M\} = \{\beta \ll l(M)\}$

(iii) $\{\alpha_1, \ldots, \alpha_k\} = \{\mathrm{Kdim}\, R/\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}\, M\}$

Proof (i). Using 2.8 and the previous lemma we see that M contains submodules N_i such that $l(N_i) = \omega^{\alpha_i} n_i$ for $1 \leq i \leq k$. Put $N := \Sigma_{i=1}^{k} N_i$. Using 2.11 in combination with the principle of coefficientwise comparison (2.10) one easily shows that this sum is direct and that $l(N) = l(M)$. The last relation shows that M is an essential extension of N .

(ii). With the notation introduced in 2.9 we are going to show $lS(M) = C(M)$. Since the inclusion $\subseteq$ was established in 2.9 we need only take care of the opposite inclusion. Let β be an arbitrary ordinal such that $\beta \ll l(M)$. We can write

$$\beta = \omega^{\alpha_k} b_k + \ldots + \omega^{\alpha_1} b_1$$

where $b_i \leq n_i$ for $1 \leq i \leq k$. By 3.1 we can find submodules $L_i \subseteq N_i$ such that $l(L_i) = \omega^{\alpha_k} b_k$. Put $L := \Sigma_{i=1}^{k} L_i$. Clearly this sum is direct, so by 2.11 we obtain $l(L) = \beta$.

(iii). We shall first prove the inclusion $\subseteq$. Let α be one of the members in the set $\{\alpha_1, \ldots, \alpha_k\}$. By (possibly repeated) application of the $_*$-operation in 2.8 to M , we obtain a submodule $N \subseteq M$ with Krull dimension α . Hence there is a prime ideal $\mathfrak{p} \in \mathrm{Ass}\, N \subseteq \mathrm{Ass}\, M$ such that $\mathrm{Kdim}\, R/\mathfrak{p} = \alpha$. Conversely, let $\mathfrak{p}$ be a prime ideal in Ass M such that Kdim $R/\mathfrak{p} = \alpha$. Then M contains an isomorphic copy of $R/\mathfrak{p}$ having length equal to ω^{α} . By the principle of coefficientwise comparision (2.10), α is one of the exponents $\alpha_1,\ldots,\alpha_k$ in the Cantor normal form of $l(M)$. ∎

3.3 Definition As in [1] we let $o(M)$ denote the supremum of the ordinal types of descending chains of non-zero submodules of M .

We close this section by expressing $o(M)$ in terms of $l(M)$.

3.4 Theorem Let M be Noetherian module over a commutative ring. Then we have

$$o(M) = \min(\omega_1, l(M)) ,$$

where ω_1 denotes the first non-countable ordinal.

Proof Let us first treat the case where $l(M) < \omega_1$. In this case Kdim M is countable. It follows from 2.12 that $o(M) \geq l(M)$. On the other hand it is easily seen that we (in general) have $o(M) \leq l(M)$. Hence

$$o(M) = l(M)$$

which proves the theorem in this case.

Let us now treat the case where $l(M) \geq \omega_1$. Let β be an arbitrary ordinal less than ω_1 . By 2.3 there exists a submodule $N_\beta \subset M$ such that $l(M/N_\beta) = \beta$. By 2.12 M/N_β has a descending chain of non-zero modules of ordinal type β , hence such a chain also exists in M . This gives $o(M) \geq \beta$, so $o(M) \geq \omega_1$. On the other hand, by 1.1 in [1] every chain in M is countable, so

$o(M) \leq \omega_1$. This gives

$$o(M) = \omega_1$$

and the proof is now complete. ∎

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